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A new approach to the calculation of the central charge

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Abstract. We show that the central charge c of the Virasoro algebra is determined by the spectrum of the Hamiltonian $L_0 + \bar{L}_0$ corresponding to a partition function which is invariant under the subgroup of modular transformations generated by S. Using this result we discuss in detail a new possibility of determining c for a given system at criticality which turns out to give excellent estimates even if the lattices accessible to numerical calculation are very small. This enables us to predict the central charge of some spin systems. Furthermore our approach to the determination of c leads to new universal functions interpolating between criticality and off-criticality.

1. Introduction

In two-dimensional conformally invariant systems the central charge c of the Virasoro algebra plays a central role in the understanding and classification of statistical mechanics models. At criticality, each scaling field corresponds to a representation of this algebra and for a given value of c the anomalous dimensions (critical exponents) x of the scaling fields and correlation functions for many classes of systems can be computed [1]. Clearly, knowledge of the central charge for a specific system is of great importance and much effort has been devoted to its determination [2–4].

In addition to that, in a statistical mechanics model defined on a torus modular invariance implies strong constraints on the possible operator content of the theory [5] and using modular transformations one can derive partition functions for various non-periodic boundary conditions imposed on the system [6]. For systems with central charge $c < 1$ only very few modular invariants exist; in the presence of higher infinite-dimensional symmetries the situation is similar even if $c \geqslant 1$. Thus, for a given value of c, the operator content of the model under consideration is almost completely fixed.

Here we ask the reversed question of whether it is possible to find the central charge if the operator content, characterized by the set of all critical exponents, is known. Several positive answers to this question have already been given ([3, 7], see later). Using modular invariance, or, to be more precise, invariance under the subgroup of modular transformations generated by S (see section 2), we will give another answer leading to a new method of determining c, i.e. we present a new relation expressing c in terms of the finite-size scaling spectrum of the Hamiltonian corresponding to an S-invariant critical system; this is the content of the virial theorem (1.12) and (2.9) discussed in section 2. It turns out that this way of computing the central charge, when applied to systems where only numerical data for finite lattices are available, leads to excellent estimates for c. Therefore, in a next step, a finite-size

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scaling analysis shows how to obtain the central charge from finite-size data. This procedure is to be discussed in detail for two reasons. Since in such a determination of the central charge one has to rely on 'experimental' data (more frequently obtained by computer than by real experiments), one has to discuss possible sources of errors. Furthermore, the finite-size scaling analysis in certain geometries [8] has already been proven to be a powerful tool in determining critical quantities such as the central charge or critical exponents and one would like to compare the results obtained from these methods.

Let us briefly remind the reader how the standard ways of determining c in a concrete system work. If some critical exponents have been measured, the simplest possible way to determine the central charge is to compare them with the prediction from the postulated conformal field theory with central charge c. However, without additional information (e.g. on some higher infinite-dimensional symmetries present in the system) this method seems to be useful only for unitary models with $c < 1$, because for $c \geqslant 1$ there are no restrictions on the allowed values of the various critical exponents . Another possibility is using the sum rule [7]

$$c = \frac{12}{n}\left(\sum_{i=1}^{n} x_i\right) - 2(n-1) + \frac{4l}{n} \tag{1.1}$$

where the x_i are the critical exponents of the n primary fields present in the system and the non-negative integer l ($l \neq 1$) characterizes the corresponding field theory. Here it is crucial to know the number of primary fields n. If l is not known, this relation gives a lower bound for the central charge in terms of the critical exponents, $\sum x_i \leqslant n(n-1)/6 + nc/12$. Note that the critical exponents with large values of x give the most significant contribution to the sum in (1.1). Usually, in finite-size calculations, these exponents are the hardest to determine.

A direct measurement of the central charge can be obtained from the ground state energy or the specific heat of the one-dimensional quantum system corresponding to the two-dimensional critical system [2, 3]. The idea is that instead of considering the infinite two-dimensional critical system to study the same system at criticality with one of the two space directions kept finite. We will consider systems of dimension $\beta \times L$ and consider L as the space direction and β as the Euclidean time direction.

Consider the partition function of a one-dimensional quantum system of length L and ground state energy E_0 at temperature $T = 1/\beta$:

$$Z = \operatorname{Tr} \mathrm{e}^{-\beta H} = \mathrm{e}^{-\beta E_0} \operatorname{Tr} \mathrm{e}^{-\beta(H-E_0)} = \mathrm{e}^{-\beta E_0} Z_{\mathrm{exc}}. \tag{1.2}$$

In the limit $\beta \to \infty$ this expression is the partition function of a two-dimensional system mapped to a strip of infinite length and *finite width* L (a cylinder in the case of periodic boundary conditions in space direction L). At criticality for L large (in the scaling limit), $H - E_0$ is related to the dilatation generator $L_0 + \overline{L}_0$ of the Virasoro algebras with central charge c [9]:

$$H - E_0 \to \frac{2\pi}{\xi L}\left(L_0 + \overline{L}_0 - \frac{c}{12}\right). \tag{1.3}$$

Here the 'sound velocity' ξ fixes the Euclidean time scale. The critical exponents x are the eigenvalues of $L_0 + \overline{L}_0$ given by the energy gaps of H. They scale as [10]

$$E_n(L) - E_0(L) = \frac{2\pi}{\xi L}(x_n + \cdots) \tag{1.4}$$

where in the limit of large L the corrections to finite-size scaling, symbolized by the dots, vanish.

We first consider a quantum system of *finite length* at zero temperature $\beta = 1/T \to \infty$, ($\beta \gg L$). Assuming periodic boundary conditions in the space direction the free energy per unit length at zero temperature $F/L = \lim_{\beta\to\infty} \beta^{-1} \ln Z/L$ is just equal to the ground state energy $-E_0/L$ per site of H. Expansion in powers of L at $T = 0$ gives [2]

$$\lim_{\beta\to\infty} F/L = -\frac{E_0(L)}{L} = \xi^{-1}A + \frac{\pi c}{\xi 6 L^2} + \cdots. \tag{1.5}$$

Here A is the non-universal bulk free energy and the dots denote (again non-universal) corrections to finite-size scaling vanishing in the scaling limit. Note that boundary conditions in the time direction are immaterial since we have taken the limit $\beta \to \infty$. This method of computing the central charge is usually used in numerical finite-size calculations. By computing E_0 as a function of L one can obtain estimates for the central charge using (1.5). In order to improve the series of estimates obtained for different values of L, one may apply some extrapolation algorithm to this series and get a final estimate for the central charge. The reliability of this estimate depends strongly on the order of magnitude of the corrections to finite-size scaling.

Note that the Hamiltonian is fixed only up to a non-universal numerical constant factor, the sound velocity ξ. In order to preserve conformal invariance, the Hamiltonian has to be normalized by this factor ξ which in most cases has to be determined approximately from the finite-size data (see [11] and section 4 where an independent means of determining ξ is discussed). This gives rise to an additional error in the determination of c. This difficulty can be avoided by considering the logarithms of the eigenvalues $t_n(L)$ of the isotropic transfer matrix $\mathsf{T}(L)$. Because of the universality of the exponents x_n and the central charge c one has

$$\frac{\ln t_0(L)}{L} = \tilde{A} + \frac{\pi c}{6L^2} + \cdots \tag{1.6}$$

and

$$-\lim_{L\to\infty} \frac{L}{2\pi} (\ln t_n(L) - \ln t_0(L)) = x_n \tag{1.7}$$

where t_0 denotes the largest eigenvalue of T. Working in the transfer matrix formalism has the advantage of avoiding problems with the normalization, but has the drawback that the finite-size corrections are usually larger [11].

Relations (1.5) and (1.4) (or (1.6) and (1.7) respectively) in many cases provide a very accurate way of measuring the central charge c and the critical exponents x_n in a given system [2, 13] and have become a standard method in analytical and numerical finite-size computations. Errors in the numerical determination of c from the ground state energy come from the generally unknown constant A ($\tilde{A}$), from the normalization ξ and from corrections to finite-size scaling.

A different method of computing c is founded on the interpretation of (1.2) as the partition function of a one-dimensional quantum system of *infinite length* L at *finite temperature* $T = 1/\beta$, i.e. we consider the case $L \gg \beta$, β *finite*. After performing the limit $L \to \infty$ one expands the free energy per unit length $T \ln Z/L$ in powers

of T and studies the low-temperature behaviour of the system. Assuming periodic boundary conditions in *time direction* (β) one finds [3]

$$\lim_{L\to\infty}\frac{F}{L}=\xi^{-1}A+\lim_{L\to\infty}\beta^{-1}\frac{\ln Z_{\mathrm{exc}}}{L}=\xi^{-1}A+\frac{\pi c}{6\xi\beta^2}+\cdots \tag{1.8}$$

where Z_{exc} (1.2) is the part of the partition function resulting from the excitation energy. This relation allows the computation of the central charge from the excitation spectrum of the Hamiltonian: Considering the thermodynamic average of the excitation energy per unit length $U_{\mathrm{exc}}/L=\partial/\partial\beta\ln Z_{\mathrm{exc}}/L$ at temperature T one obtains

$$\lim_{L\to\infty}\frac{U_{\mathrm{exc}}}{L}=-\frac{\pi c}{6\xi\beta^2}+\cdots. \tag{1.9}$$

Assuming $\xi=1$ and using (1.4) this gives

$$c=12\lim_{L\to\infty}\frac{\beta^2}{L^2}\frac{\sum_n x_n\mathrm{e}^{-2\pi x_n\beta/L}}{\sum_n\mathrm{e}^{-2\pi x_n\beta/L}}+\cdots. \tag{1.10}$$

Now taking the low temperature limit $\beta=\delta L$ and keeping in mind that throughout this discussion $L\gg\beta$ was assumed one obtains

$$\begin{aligned}c&=12\lim_{\delta\to0}\delta^2\frac{\sum_n x_n\mathrm{e}^{-2\pi x_n\delta}}{\sum_n\mathrm{e}^{-2\pi x_n\delta}}\\&=12\lim_{\delta\to0}\delta^2\langle L_0+\overline{L}_0\rangle_\delta\end{aligned} \tag{1.11}$$

expressing the central charge in terms of the scaled energy gaps of H. Another expression can be obtained from the low temperature limit of the specific heat per unit length by taking the second derivative of F/L with respect to T. This way of measuring the central charge has been used in many cases where the specific heat could be computed analytically and in several experiments [3, 12]. Note that in numerical applications this technique of determining c has the drawback that two approximations are necessary. First data have to be extrapolated to $L\to\infty$ to obtain (1.11) or the specific heat per unit length $c_V(T)=\lim_{L\to\infty}C_V(L,T)/L$. Then, out of this function, the central charge has to be extracted.

So far this short review. We want to stress that in reading formulae (1.8)–(1.11) it is essential that the limit $L\to\infty$ has already been performed, (i.e. one assumes $L\gg\beta$), while in (1.5) the limit $\beta\to\infty$ was considered ($\beta\gg L$). In this paper we consider the case when both β and L are taken to infinity, the ratio $\beta/L=\delta$ kept at fixed *finite* value. As in (1.11), in our approach c turns out to be given by the spectrum of $L_0+\overline{L}_0$, i.e. c is determined by the scaled (and normalized) energy gaps of the Hamiltonian (or the transfer matrix respectively) rather than by the ground state of H only. However, the expression derived later (see equation (2.9)) is different. For certain choices of boundary conditions in space and time direction we obtain the virial theorem ($\xi=1$)

$$c=12\frac{\delta\langle L_0+\overline{L}_0\rangle_\delta+\frac{1}{\delta}\langle L_0+\overline{L}_0\rangle_{\frac{1}{\delta}}}{\delta+\frac{1}{\delta}}. \tag{1.12}$$

For $\delta \to 0$ (low temperature limit $L \gg \beta$) we recover Affleck's result (1.11). In numerical applications we will focus on the case $\delta = 1$ (corresponding to a system defined on a quadratic torus of dimension $L \times L$, $L \to \infty$). It turns out that in such a geometry the virial theorem (1.12) can be used to obtain finite-size scaling estimates for the central charge converging very well to the true value of c in the infinite system. The reason why we choose $\delta = 1$ is that, because of the symmetry of (1.12) in δ and $1/\delta$, the contribution of errors in finite-size estimates of the scaled energy gaps of the Hamiltonian (1.2) (eigenvalues of $L_0 + \bar{L}_0$) to the expression (1.12) is minimal at this particular value. Two advantages of our result compared with the determination of c from the ground state energy are a reduced sensitivity to errors in the normalization and the fact that the non-universal constant A does not appear in our equations. Compared with the determination from the energy U_{exc}/L an obvious advantage is that only one extrapolation is necessary and no additional fit. There is also the advantage of a reduced sensitivity in the dependence on the normalization.

Studying the expression given in equations (1.12) or (2.9) respectively, one finds that in numerical applications there are two different strategies in determining c. The first is to extrapolate the critical exponents from the finite-size data and then to use these extrapolants to obtain an estimate for c. The second is to take the scaled finite-size energy gaps as they are and get an estimate $c(L)$ depending on the lattice size and then to extrapolate $c(L)$. We will focus on the latter strategy since it can be applied more generally. Many examples presented in section 3 show that good estimates for c can be obtained even from tiny lattices. Combining the various methods of determining the central charge described earlier will improve the numerical accuracy considerably and give (together with the critical exponents) valuable information on the universality class the system belongs to, even if $c \geqslant 1$.

We want to stress that so far all parameters on which the Hamiltonian depends (like temperature or magnetic field strength) were assumed to be such that the corresponding two-dimensional system is critical†. However, it is interesting to note that by expressing the critical exponents in terms of the scaled energy gaps as functions of the parameters of the two-dimensional system their relation to the central charge extends away from criticality into the scaling region and thus defines new universal functions of temperature, magnetic field strength, etc characterizing the systems under consideration. At the critical point, these functions take the value c, which is the central charge. This is going to be discussed in the case of the Ising model with a thermal perturbation, where we can compute this function exactly.

The paper is organized as follows. In the following section we define a class of universal functions $C_{BB'}$ depending on the same parameters $\{g\}$ as the Hamiltonian (transfer matrix) and study their behaviour in the critical ($\{g\} = \{g_c\}$) range of parameters. In particular, we show that $C_{BB'}(g_c) = c$. In section 3 we discuss the finite-size corrections to $C_{BB'}$. In the critical region $C_{BB'}$ turns out to converge very rapidly to c in many cases, providing excellent estimates for c even if only data for very small lattices are available. In a comparison in some examples this method of determining c gives better results than the standard way of estimating c from the ground state energy described earlier. We present results for well known models such as the Ising model and conjectures for cases, where c is not yet known. In section 4

† In particular, the temperature variable of the two-dimensional system used in section 5 should not be confused with the temperature $T = 1/\beta$ of the one-dimensional system.

the dependence of the measurement of c on the normalization ξ is studied. As a by-product of our considerations we propose a way of determining the normalization ξ of the Hamiltonian using modular invariance. In section 5 we briefly comment on the properties of the universal functions $C_{BB'}(g)$ at off-criticality. As an example we consider the Ising model with a thermal perturbation in zero magnetic field. In the last section we summarize and discuss our results.

2. The universal $C_{BB'}$ functions at criticality

Let the Hamiltonian (1.2) of a system of finite length L depend on a set of parameters $\{g\}$ such as temperature or magnetic field strength. In the sequel this set of parameters $\{g\}$ will simply be denoted by g. We denote by $E_0(g; L)$ the ground state energy of the Hamiltonian H. Consider the limit $L, \beta \to \infty$, with

$$\beta = \delta L \qquad \delta = \text{constant}. \tag{2.1}$$

The partition function on a rectangle of dimension $\beta \times L$ with periodic boundary conditions, i.e. a torus, is given by [10]

$$Z_{PP}(L, \beta) = \operatorname{Tr} e^{-\beta H} = \operatorname{Tr} e^{(-LH)\delta} \tag{2.2}$$

where $\delta = \beta / L$ is the modular parameter of the torus. Define the universal function $C_{PP}(\{g\})$ for the quadratic torus with $\delta = 1$ by

$$C_{PP}(g) = \frac{6}{\pi} \lim_{L\to\infty} \left[-\frac{\partial}{\partial \delta} \ln Z_{PP}(g; L; \delta) - L \cdot E_0(g; L) \right]_{\delta=1}. \tag{2.3}$$

Inserting (2.2) one gets

$$\begin{aligned} C_{PP} &= 12 \lim_{L\to\infty} \frac{L}{2\pi} \frac{\operatorname{Tr}(H - E_0) e^{-LH}}{\operatorname{Tr} e^{-LH}} \\ &= 12 \lim_{L\to\infty} \frac{\sum_n L/2\pi (E_n - E_0) \exp\{-2\pi [L/2\pi (E_n - E_0)]\}}{\sum_n \exp\{-2\pi [L/2\pi (E_n - E_0)]\}} \\ &= 12 \lim_{L\to\infty} \frac{L}{2\pi} \langle H - E_0 \rangle_{\delta=1}. \end{aligned} \tag{2.4}$$

This function, the mean value of the scaled energy gaps on the quadratic torus (in one-dimensional language: the thermodynamic average of the scaled excitation energy at inverse temperature $\beta = L$)†, is the basic object we are dealing with in this paper. Its properties and generalization to other boundary conditions are going to be discussed here and in the following sections.

Suppose that the system undergoes a second-order phase transition for a certain set of coupling constants $\{g\} = \{g_c\}$. At criticality in a conformally invariant system Z_{PP} is modular invariant [6], so in particular it is invariant under the subgroup of modular transformations generated by $S : \delta \to 1/\delta$. In the large L limit we have

$$Z_{PP}(g_c; \delta) = \operatorname{Tr} \exp\left[-2\pi \left(L_0 + \bar{L}_0 - \frac{c}{12} \right) \delta \right] = Z_{PP}\left(g_c; \frac{1}{\delta} \right) \tag{2.5}$$

† Not $L \gg \beta$ as in (1.11)!

where (1.3) was used (we assume H to be normalized). Thus

$$\frac{\partial}{\partial\delta}Z_{PP}(g_c;\delta) = -\frac{1}{\delta^2}\left(\frac{\partial}{\partial(1/\delta)}Z_{PP}(g_c;1/\delta)\right). \tag{2.6}$$

From (2.5) and (2.6) follows

$$\frac{c}{12} - \langle L_0 + \overline{L}_0\rangle_\delta = -\frac{1}{\delta^2}\left(\frac{c}{12} - \langle L_0 + \overline{L}_0\rangle_{1/\delta}\right) \tag{2.7}$$

which is the result quoted in the introduction (1.12). In particular for $\delta = 1$ we can formulate the virial theorem as

$$c = 12\langle L_0 + \overline{L}_0\rangle_{\delta=1}. \tag{2.8}$$

Inserting the scaling properties (1.3) and (1.4) into the definition of $C_{PP}(g)$ (2.3) we find $Z_{PP}(g_c;\delta) = e^{\pi c/6}\sum_n e^{-2\pi x_n\delta}$ and the virial theorem expressed in terms of the function C_{PP} reads as

$$\begin{aligned} C_{PP}(g_c) &= 12\langle L_0 + \overline{L}_0\rangle_{\delta=1} \\ &= 12\frac{\sum_n x_n e^{-2\pi x_n}}{\sum_n e^{-2\pi x_n}} \\ &= c. \end{aligned} \tag{2.9}$$

Since in our approach we consider the $\beta, L \to \infty$ with condition (2.1), δ finite, boundary conditions in both *space and time* direction, are relevant. The central charge of the Virasoro algebra can be computed from the set of anomalous dimensions of the scaling operators and their descendents belonging to an S-invariant partition function. Rephrased in a different way, this means that c is determined by the spectrum of $L_0 + \overline{L}_0$ given by the energy gaps of the Hamiltonian (or transfer matrix) describing the system. We want to stress that the sum in (2.9) does not only contain the anomalous dimensions of the primary fields but runs over the complete spectrum of $L_0 + \overline{L}_0$.

Instead of using Z_{PP} one may insert into the definition of C (2.3) other linear independent S-invariant partition functions $Z_{BB'}$ corresponding to systems with boundary conditions B, B' in space and time direction defining new functions $C_{BB'}(g)$.

$$C_{BB'}(g) = \frac{6}{\pi}\lim_{L\to\infty}\left[-\frac{\partial}{\partial\delta}\ln Z_{BB'}(g;L;\delta) - L\cdot E_0(g;L)\right]_{\delta=1}. \tag{2.10}$$

Because of S-invariance

$$\left(\frac{\partial}{\partial\delta}Z_{BB'}(g_c;\delta)\right)_{\delta=1} = 0 \tag{2.11}$$

thus

$$C_{BB'}(g_c) = c. \tag{2.12}$$

Note that this holds true for any linear combination of S-invariant partition functions $Z_\alpha = \sum_{i=1}^m \alpha_i Z_{B_iB_i'}$. This defines functions $C_\alpha(g)$ corresponding to different choices of boundary conditions specified by the set of numbers α. At criticality they satisfy

$$C_\alpha(g_c) = c \tag{2.13}$$

independent of α. Each of the functions C_α gives an independent determination of the central charge c. Equations (2.9), (2.12) and (2.13) result merely from S-invariance of the partition function and therefore apply to any two-dimensional conformally invariant system.

3. Application to finite systems

So far, we have discussed results which are valid for infinite systems. In order to compute the central charge of a specific system exactly one either has to calculate the free energy per site in a strip of finite width L at zero temperature and expand it in powers of L^{-1} (see (1.5)) or one computes the free energy per site of the infinite system at finite temperature T and expands it in powers of T (1.8). The virial theorem (2.9) opens the new possibility of computing c of a system of length L and temperature $T = 1/L$ in the limit $L \to \infty$, $LT = 1$. As we discuss in this section, this can be used to obtain excellent estimates for the central charge in cases where only numerical data for finite systems are available. In such systems only the low-lying part of the spectrum can be determined and consequently not the complete (infinite) set of excitations of the primary fields. Apart from that the scaled energy gaps $\mathcal{E}_n(L) = \xi(L/2\pi)(E_n - E_0)$ (see 1.4) differ from the exact values of the critical exponents x_n by a finite-size correction ε_n. Both effects have to be taken into account when trying to extract an estimate for c from finite-size data.

3.1. *Effect of cutting the Hilbert space of states*

Before we study finite-size effects, it is important to note that the sum in the right-hand side of (2.9) is very rapidly converging, therefore for an approximative computation of c it is possible to restrict the infinite set of critical exponents to a small number of low-lying excitations in the spectrum. This is important since in finite-size calculations only the lower part of the spectrum can be determined. It is worth studying the effect of cutting the space of states on the computation of c by means of (2.9) in concrete examples.

Consider the Ising model in zero magnetic field on a two-dimensional lattice of dimension $\beta \times L$. The partition function is

$$Z = \sum_{\text{config.}} \exp\left(-k \sum_{\text{NN}} \sigma_i \sigma_j\right) \tag{3.1}$$

where k is the inverse of the temperature (not to be confused with β, the temperature of the one-dimensional quantum system). The sum inside the exponential runs over all nearest neighbours in the lattice and the sum outside over all configurations of spins $\sigma_i = \pm 1$. The infinite system is critical if $k = k_c = \frac{1}{2}\ln(1+\sqrt{2})$ with central charge $c = \frac{1}{2}$. According to (2.5) at criticality the partition function on a torus of dimension $L \times \beta$ can be written as

$$Z(k_c; \delta) = \operatorname{Tr} e^{-2\pi(L_0 + \overline{L}_0 - 1/24)\delta}. \tag{3.2}$$

The (infinite-dimensional) space of irreducible representations of the Virasoro algebra is spanned by the eigenstates of the (scaled) Hamiltonian $L_0 + \overline{L}_0$ with eigenvalues $x = \Delta + r + \overline{\Delta} + \overline{r}$. The highest weight representations ($r = \overline{r} = 0$) correspond to the scaling fields with critical exponents $x = \Delta + \overline{\Delta}$. For periodic boundary conditions this are the primary fields energy density ϵ with dimension $(\Delta, \overline{\Delta}) = (\frac{1}{2}, \frac{1}{2})$, $x_\epsilon = 1$, the magnetization σ ($(\Delta, \overline{\Delta}) = (\frac{1}{16}, \frac{1}{16})$, $x_\sigma = \frac{1}{8}$) and the identity operator 1 with $(\Delta, \overline{\Delta}) = (0, 0)$, $x = 0$. In terms of characters of the Virasoro algebra the partition function reads

$$Z_{PP}(\delta) = \chi_0 \overline{\chi}_0 + \chi_{1/2} \overline{\chi}_{1/2} + \chi_{1/16} \overline{\chi}_{1/16} \tag{3.3}$$

where

$$\chi_0(\delta) = \tfrac{1}{2}z^{-1/48}\left[\prod_{n=1}^{\infty}\left(1+z^{n-1/2}\right)+\prod_{n=1}^{\infty}\left(1-z^{n-1/2}\right)\right]$$

$$\chi_{1/2}(\delta) = \tfrac{1}{2}z^{-1/48}\left[\prod_{n=1}^{\infty}\left(1+z^{n-1/2}\right)-\prod_{n=1}^{\infty}\left(1-z^{n-1/2}\right)\right] \tag{3.4}$$

$$\chi_{1/16}(\delta) = z^{-1/48}\left[z^{1/16}\prod_{n=1}^{\infty}(1+z^n)\right]$$

and $z = \mathrm{e}^{-2\pi\delta}$. It turns out that ignoring all but the primary fields, i.e. restricting the space of Virasoro states to the irreducible highest weight representations, gives a good approximation for c. In the case of the Ising model inserting the dimensions $x = 0, x_\sigma = \frac{1}{8}$ and $x_\epsilon = 1$ into (2.9) and neglecting the contribution from the descendents $(r, \bar{r} \geqslant 1)$ yields $c \approx 12(\frac{1}{8}\exp -2\pi/8 + \exp -2\pi)/(1+\exp -2\pi/8 + \exp -2\pi) \approx 0.485$.

Including the lowest lying excitations with $r + \bar{r} \leqslant 1$ $(x_n \leqslant 2)$ the result is $c \approx 0.4998$ and taking all descendents with $r+\bar{r} \leqslant 2$ $(x_n \leqslant 3)$ leads to $c \approx 0.499\,94$ (see table 1).

Table 1. Different approximative values for the central charge c computed from (2.9) for the Ising model (rows 1 and 2), the three-state Potts model (row 3) and the four-state Potts model (row 4) using partition functions Z_{PP} corresponding to periodic boundary conditions and $\tilde{Z}$ (see (3.6)). The calculation is done using critical exponents with $x_n \leqslant$ 1, 2 and 3 respectively.

Z	$x_n \leqslant 1$	$x_n \leqslant 2$	$x_n \leqslant 3$	c
Z_{PP}	0.485	0.4998	0.499 94	0.5
$\tilde{Z}$	0.497	0.499 98	0.499 992	
Z_{PP}	0.773	0.7998	0.799 999	0.8
Z_{PP}	0.959	0.9995	0.999 90	1

There is one linear independent S-invariant partition function Z_{AA} corresponding to antiperiodic boundary conditions in space and time direction

$$Z_{AA} = \chi_{1/16}\bar{\chi}_{1/16} - \chi_0\bar{\chi}_{1/2} - \chi_{1/2}\bar{\chi}_0. \tag{3.5}$$

With Z_{AA} one can construct the S-invariant linear combination $\tilde{Z} = Z_{PP} - Z_{AA} = Z_{AP} + Z_{PA}$ which in terms of Virasoro characters is given by

$$\tilde{Z} = \left|\chi_0 + \chi_{1/2}\right|^2 = z^{1/24}\prod_{n=1}^{\infty}\left(1+z^{n-1/2}\right)^2 \tag{3.6}$$

The primary fields appearing in $\tilde{Z}$ are the identity operator with dimension $(0,0)$, the Mayorana fermions Ψ and $\bar{\Psi}$ with dimensions $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ respectively and the energy density ϵ with $(\Delta,\bar{\Delta}) = (\frac{1}{2},\frac{1}{2})$. As already seen very few exponents lead to almost exact values for c. This holds true not only for the Ising model but

is also a general feature of expression (2.9) making it useful for determining the central charge. In table 1 we list approximate values for the central charge of the Ising model obtained from Z_{PP} and $\tilde{Z}$ and of the three- and four-state Potts model obtained from Z_{PP} for various cutoffs in the space of irreducible representations of the Virasoro algebra.

3.2. *Finite-size corrections*

As discussed in section 3.1, we do not expect the lack of higher excitations in the finite-size spectrum to be an essential obstacle to an approximative calculation of c. Excitations with large x do not give significant contributions to the sums in (2.9) and can be ignored. On the other hand, the finite-size contributions to x_n lead to finite-size corrections to c. This has to be discussed in more detail. Here as throughout this section we assume the Hamiltonian to be normalized properly.

First of all, as mentioned in the introduction, there are two strategies one can follow. One possibility is first to obtain estimates for critical exponents by an extrapolation of the scaled energy gaps to $L \to \infty$ and then to use (2.9) to calculate an estimate $c_{\exp}$. The difference ε_n between the estimated exponents $x_n^{(\exp)}$ and the correct values x_n leads to an error in c as discussed later. If the low-lying part of the spectrum has been determined completely, it is given up to first order in the differences ε by (3.9) with $C(L)$ replaced by $c_{\exp}$. On the other hand, in the system under consideration, it might be impossible to obtain all critical exponents which give relevant contributions to (2.9). In such a case the resulting error in the determination of c would be large.

The second method is first to insert the complete set of finite-size data, the scaled energy gaps $\mathcal{E}_n(L)$ into (2.9). This defines a function $C(L)$ converging to c. Now this function can be extrapolated in order to improve the result. In order to decide which extrapolation algorithm has to be applied and how reliable the result is, one must have some knowledge of the order of magnitude of the finite-size correction to c and the scaling properties of $C(L)$ which can be derived from the scaling behaviour of the energy gaps.

We assume the scaled energy gaps $\mathcal{E}_n(L)$ to be given by [6]

$$\begin{aligned}\mathcal{E}_n(L) &= \frac{L}{2\pi}(E_n - E_0) \\ &= x_n + \alpha_n^{(1)} L^{-y_n^{(1)}} + \alpha_n^{(2)} L^{-y_n^{(2)}} + \cdots \\ &= x_n + \varepsilon_n .\end{aligned} \tag{3.7}$$

Here the $\alpha_n^{(i)}$ are constants and $y_n^{(i)} > 0$. $\varepsilon_n = \varepsilon_n(L)$ denotes the finite-size correction to the exponent x_n. Then $C(L) \equiv C(g_c; L)$ for a finite lattice as a function of L is defined by

$$\begin{aligned}C(L) &= 12\frac{\sum_n \mathcal{E}_n(L)\exp(-2\pi\mathcal{E}_n(L))}{\sum_n \exp(-2\pi\mathcal{E}_n(L))} \\ &= 12\frac{\sum_n (x_n + \varepsilon_n)\exp(-2\pi\varepsilon_n)\exp(-2\pi x_n)}{\sum_n \exp(-2\pi\varepsilon_n)\exp(-2\pi x_n)} \\ &= 12\frac{\langle x\exp(-2\pi\varepsilon)\rangle + \langle \varepsilon\exp(-2\pi\varepsilon)\rangle}{\langle\exp(-2\pi\varepsilon)\rangle}.\end{aligned} \tag{3.8}$$

Expansion in ε leads to the scaling behaviour of the universal C-function at the critical point. Up to first order in ε the finite-size correction to C is given by

$$\begin{aligned} C(L) &\approx c + (12 + 2\pi c)\langle\varepsilon\rangle - 24\pi\langle x\varepsilon\rangle \\ &= c + a_1 L^{-y_1} + a_2 L^{-y_2} + \cdots \end{aligned} \tag{3.9}$$

with constants a_k and exponents y_k determined by (3.7). The (infinite) set of exponents y_k appearing in this expansion is the same as the set $\{y_n^{(i)}\}$. The order of magnitude of the finite-size correction to c is that of the finite-size corrections to the critical exponents. The powerlike convergence suggests the application of the algorithm of Bulirsch and Stoer [15] for an extrapolation to $L \to \infty$.

3.3. Determination of c in some specific systems

In order to test the reliability of computing the central charge from the virial theorem (2.9) using finite-size data, we have checked it in many cases where the central charge is known. After having convinced ourselves that this method produces reliable results we applied it to some systems, where it is not yet known. We discuss some examples. In table 2 we present the function $C(L)$ and extrapolated values for various models where c is known.

Table 2. Different approximative values $C(L)$ for the central charge $c = C(\infty)$ computed from quantum Hamiltonians with L sites for the Ising model, the three-state Potts model and the Ashkin–Teller model [16] for various values of the coupling constant. $h = \frac{1}{4}$ corresponds to the four-state Potts model and $h = \frac{1}{3}$ to the point with Zamolodchikov–Fateev symmetry [17]. At $h = \frac{1}{2}$ the system reduces to two decoupled Ising models and at $h = 1$ a Kosterlitz–Thouless phase transition takes place. c_{exp} is an extrapolated quantity while c_{theor} denotes the actual value of the central charge.

			Ashkin–Teller			
L	Ising	Potts(3)	$h = \frac{1}{4}$	$h = \frac{1}{3}$	$h = \frac{1}{2}$	$h = 1$
1	0.5136	0.8526	1.1100	1.1055	1.0272	0.6661
2	0.5541	0.8892	1.1222	1.1202	1.1083	1.0034
3	0.5339	0.8519	1.0672	1.0644	1.0677	1.0249
4	0.5184	0.8267	1.0337	1.0335	1.0368	1.0145
5	0.5108	0.8146	1.0177	1.0178	1.0216	1.0074
c_{exp}	0.500 00(2)	0.8000(1)	1.00(1)	0.995(7)	1.004(4)	0.997(5)
c_{theor}	0.5	0.8	1	1	1	1

The extrapolated values computed using (3.8) are in excellent agreement with the predictions. In addition we find that in many cases the finite-size data evaluated from very small lattices give surprisingly good estimates for c. The scaled energy gaps $\mathcal{E}_n(L)$ are computed from the eigenvalues of the corresponding quantum Hamiltonian with 14 sites (Ising model) and 11 sites (three-states Potts model) respectively. In the case of the Ashkin–Teller model [16] we have restricted our calculation to chains of length up to five sites in order to simulate a more typical situation. In order to get information on the error in the determination of c resulting exclusively from finite-size corrections, we have selected examples, where the normalization ξ of the

Hamiltonian is known. The error resulting from the measurement of ξ is investigated in the next section.

It is tempting to apply (3.8) to transfer matrices of small lattices as well. Here in principle no normalization has to be fixed (see introduction), only finite-size corrections contribute to the error in c. As shown in table 3 values for c computed from the eigenvalues of transfer matrices for only one site are very close to the real central charge of the infinite system! The fast convergence of $C(L)$ is particularly useful when only small lattices are accessible to numerical calculations.

Table 3. Different approximative values $C(1)$ for the central charge c of the Ising model, the three- and four-state Potts model and the six-vertex model computed using (3.8) from the eigenvalues of the corresponding transfer matrices for one site. The value obtained for the six-vertex model corresponds to the choice of parameters $\eta = \pi/12$, $v - \eta = \pi/2$ in Baxter's parametrization of the vertex weights [18].

Model	Ising	Potts(3)	Potts(4)	Six-vertex
$C(1)$	0.493	0.811	1.049	1.049
c	0.5	0.8	1	1

As an example of the strategy first mentioned, first evaluating extrapolants for the critical exponents and then calculating c from these, we consider the six-state quantum chain [19, 20]. The spectrum of the six-state quantum chain with toroidal boundary conditions is known for certain values of the coupling constants [20]. For special choices of coupling the system has Zamolodchikov–Fateev symmetry [17] with central charge $c = 1.25$. Inserting the measured exponents with $x \leqslant 2$ as given in [20] for the spectrum with periodic boundary conditions into (2.9) we obtain $c = 1.244(3)$, while the determination from the ground state energy gives $c = 1.25(1)$.

Next we want to compare the determination of the central charge discussed in this work with the traditional numerical methods, i.e. using the correction to the ground state energy per site (1.5) and from the excitation energy (1.11). For the XXZ-quantum chain (corresponding to the six-vertex model with $c = 1$) Bonner and Fisher determined numerically the low-temperature specific heat from chains of length up to 11 sites [21]. Using (1.11) there results give $c = 1.1$. On the the other hand, from (3.8) one obtains $c = 1.00(3)$ from data from up to 10 sites and extrapolating to $L \to \infty$.

When determining c from the ground state energy (1.5) one subtracts $E(L)/L - E(L-1)/(L-1)$ in order to eliminate the generally unknown constant A and obtains an estimate

$$c_{\mathrm{fe}}(L) = (L \cdot E(L) - (L-1) \cdot E(L-1)) \frac{6L(L-1)}{\pi(2L-1)}. \tag{3.10}$$

In table 4 we show finite-size estimates and BST extrapolants for c obtained from the transfer matrix of the critical Ising model (3.1) with periodic boundary conditions. We assume the bulk free energy per site A to be unknown and restrict the calculations of the estimates by (3.8) (left columns) and (1.5) (right columns) to small numbers of sites. In order to study the dependence of the extrapolated value from the maximal number of lattice sites available we calculate the extrapolants using the data $C(L)$ and $c_{\mathrm{fe}}(L)$ with $1 \leqslant L \leqslant 3$, $1 \leqslant L \leqslant 5$ and $1 \leqslant L \leqslant 8$. Comparing the finite-size

results and the extrapolants obtained from the values up to three, five and eight sites we find that the determination of c from the ground state energy is less precise than the computation from the energy gaps. The difference in precision becomes less pronounced as the number of sites increases.

Table 4. Different approximative values for the central charge c of the Ising model with L sites computed from the eigenvalues of the transfer matrix. In column 2 the estimates $C(L)$ obtained from (3.8) are shown, while column 4 contains estimates $c_{fe}(L)$ obtained from the finite-size correction to the free energy per site, see (1.5). In columns 3 and 5 BST-extrapolants of the respective series up to three, five and eight sites are given.

L	$C(L)$	c_{exp}	$c_{f.e.}(L)$	c_{exp}
1	0.4930		—	
2	0.5930		0.5676	
3	0.5658	0.55(2)	0.6034	0.64(4)
4	0.5379		0.5653	
5	0.5225	0.505(5)	0.5380	0.49(2)
6	0.5145		0.5235	
7	0.5102		0.5158	
8	0.5075	0.502(2)	0.5113	0.503(3)

Having gained confidence in the applicability of (2.9) to small lattices we consider the spin-j XX quantum chain [22]

$$H = \xi \left(\sum_{i=1}^{N} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) \tag{3.11}$$

where $S_i^{x(y)}$ denotes the spin-j matrix acting on site i. Table 5 shows data obtained from chains of length $N = 2$ with periodic boundary conditions for different j. The normalization was obtained from the partition function with antiperiodic boundary conditions (see section 4). The spin-$\frac{1}{2}$ chain is known to have $\xi = 1$ and $c = 1$. Using finite-size data of the spin-1 chain with $N = 8$ [22] one obtains $C(8) = 1.02$ while $C(2) = 1.18$. From the estimates presented in table 5 we predict that $c = 1$ independent from j.

Table 5. Different approximative values for the normalization ξ and the central charge c computed from (3.8) for the spin-j XX quantum chain with two sites. The data support the conjecture $c = 1$ independent from j.

j	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$\xi(2)$	0.88	0.57	0.43	0.34
$C(2)$	0.99	1.18	1.15	1.12
$C(8)$	—	1.02	—	—
c_{conj}	1	1	1	1

4. Dependence of c on the normalization ξ

One possible method of computing the central charge and critical exponents of a statistical mechanics system is diagonalizing a quantum Hamiltonian H corresponding

to this model. However, in order to preserve conformal invariance at the critical point, the normalization ξ of H which fixes the Euclidean time scale has to be chosen appropriately. In the previous section in most cases we considered models, where the normalization ξ of the corresponding quantum Hamiltonian was known. We denote this normalization by ξ_{crit}. In most cases, however, ξ_{crit} has to be determined from the finite-size data and we have to discuss the sensitivity of the determination of the central charge to an uncertainty in the determination of ξ_{crit}. From finite-size calculations in the Hamilton formalism ξ_{crit} can be computed only approximately. Denote this 'experimental' value by ξ_{exp} with $\xi_{\text{exp}} = \rho\xi_{\text{crit}}$, $\rho \neq 1$. The eigenvalues of H then differ from the eigenvalues of the properly normalized Hamiltonian by a factor ρ. Using the conventional method of determining c via the correction to the ground state energy (1.5) or from the excitation energy (1.11) one obtains

$$c_{\text{exp}} = \rho c \tag{4.1}$$

(In contrast to section 3, here we assume that no finite-size corrections contribute.) Setting $\alpha = (\xi_{\text{exp}} - \xi_{\text{crit}})/\xi_{\text{crit}} = \rho - 1$ we find the error Δ in the determination of c,

$$\Delta(\alpha) = (c_{\text{exp}} - c)/c = \alpha. \tag{4.2}$$

On the other hand, computing c by (2.9) gives, up to first order in α,

$$\begin{aligned} c_{\text{exp}} &= 12\frac{\sum_n (1+\alpha)x \exp[-2\pi(1+\alpha)x]}{\sum_n \exp[-2\pi(1+\alpha)x]} \\ &= c + (12 + 2\pi c)\,\alpha\frac{c}{12} - 24\pi\alpha\langle x^2\rangle \end{aligned} \tag{4.3}$$

where we haved used (3.9) with $\varepsilon_n = \alpha x_n$. S-invariance of Z gives $\langle x^2\rangle > (c/12)^2$ and defining $\gamma > 0$ by $\langle x^2\rangle = (1+\gamma)(c/12)^2$ we find

$$\Delta(\alpha) = \left(1 - \gamma\frac{\pi c}{6}\right)\alpha < \alpha \qquad \text{if } c > 0. \tag{4.4}$$

In a neighbourhood of ξ_{crit}, c_{exp} as a function of ξ_{exp} computed by (2.9) cannot grow faster as when computed from the ground state energy (1.5). In all examples we checked numerically, we even found $|\Delta(\alpha)| < \alpha$, near ξ_{crit}, i.e. c computed from the critical exponents is less sensitive to a wrong normalization compared with a determination from the ground state energy. As examples we show c as a function of ξ determined by (2.9) for the Ising model (figure 1) and determined from (3.8) for the spin-1 XX-chain with two sites (figure 2).

The correct determination of ξ_{crit} can be accomplished by normalizing the scaled energy gaps ΔE between excited states with momenta P and P' of the same primary field to $\Delta E = P - P'$ [11]. Here we propose an independent method using the modular transformation S. Consider a system on the torus with boundary conditions B and B' in space and time direction respectively. If ξ has been chosen as ξ_{crit} such that the system is conformally invariant at the critical point, then after a transformation $S : \delta \to 1/\delta$ the partition function $Z_{BB'}$ satisfies

$$Z_{BB'}\left(\frac{1}{\delta}\right) = Z_{B'B}(\delta). \tag{4.5}$$

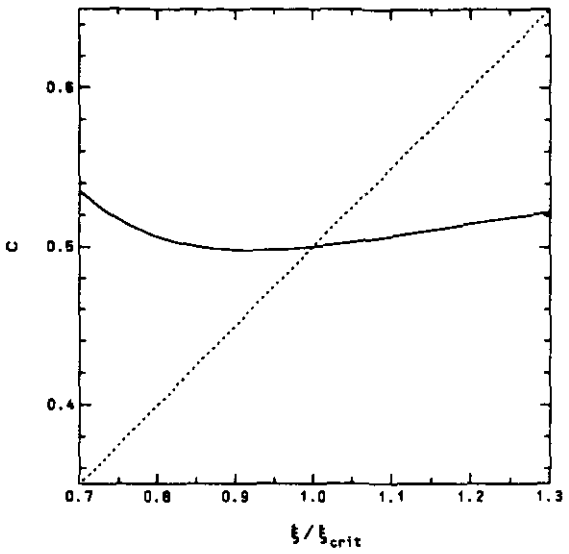

Figure 1. The central charge c of the Ising model computed from the critical exponents given by the partition function (3.3) as a function of the normalization ξ/ξ_{crit} using the virial theorem (2.9) (full curve). ξ_{crit} denotes the correct normalization of the corresponding quantum Hamiltonian. The dotted curve shows the dependence of c on ξ/ξ_{crit} when calculated from the ground state energy (1.5).

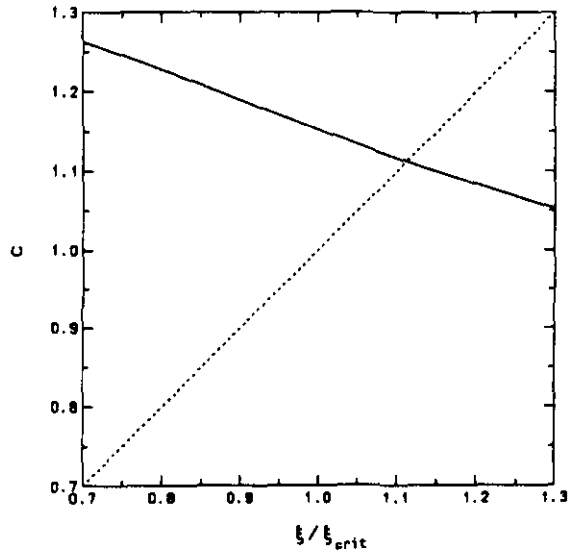

Figure 2. The central charge c of the spin 1 XX quantum chain computed from the scaled energy gaps of the Hamiltonian (3.11) with two sites as a function of the normalization ξ/ξ_{crit} using formula (3.9) (full curve). Here $\xi_{crit} \approx 0.61$ denotes the normalization of the Hamiltonian obtained approximately in [17]. The dotted curve shows the dependence of c on ξ/ξ_{crit} when calculated from the ground state energy (1.2) under the assumption that $c = 1$ (see table 4 and [17]).

In principle (4.5) can be used to determine ξ_{crit}. For a numerical determination consider the partition function on the quadratic torus ($\delta = 1$). Here

$$Z_{BB'} = Z_{B'B} \tag{4.6}$$

which is non-trivial if $B \neq B'$. In a finite system of width L this equation defines a function $\xi(L)$ converging to ξ_{crit} as $L \to \infty$. Different pairs of boundary conditions define different functions $\xi(L)$ all of them converging to the same value ξ_{crit} at the critical point.

5. The universal $C_{BB'}$ functions at off-criticality in the Ising model

The functions $C_{BB'}(g)$ are also defined at off-criticality and therefore allow for an extension of the definition of the central charge into the scaling region. Here we want to calculate these functions for the Ising model in zero magnetic field, where the energy gaps as a function of the temperature (of the two-dimensional system) are known.

First we briefly discuss the qualitative structure of the functions $C_{BB'}$ at off-criticality in the general case. Suppose the system to be in the scaling region near the critical point. The energy gaps are continuous functions of the scaling variables $\mu^{(i)}$ which we define by

$$\mu^{(i)} = \left(g^{(i)} - g_c^{(i)}\right) L^{2-x^{(i)}} \tag{5.1}$$

where $x^{(i)}$ is the critical exponent of the field corresponding to the perturbation (i) with correlation length $\xi^{(i)}$ and the ratio $\xi^{(i)}/L$ is fixed (L is the size of the system).

Consider the same partition functions $Z_\alpha = \sum_i \alpha_i Z_{B,B'_i}$ as before and the corresponding functions C_α. Clearly $C_\alpha(\mu)$ is a continuous function of the scaling variables $\{\mu^{(i)}\}$. As discussed earlier, in the critical range of coupling constants $\mu^{(i)} = 0$ these functions take the value c of the central charge independent of the boundary conditions. Far away from the scaling region all the scaled energy gaps either diverge or vanish. Then all terms in the numerator in the right-hand side of equation (2.9) become negligible while the denominator remains finite due the presence of the lowest gap which, by definition, is 0. As a consequence C vanishes independently from α corresponding to the absence of a universal correction to the ground state energy in non-critical systems (see figures 3 and 4, $\mu \to \infty$). $C_\alpha(\mu)$ continuously interpolates the central charge between the critical and non-critical range of coupling constants and is a measure of the numbers of degrees of freedom.

These properties of the C functions can be easily studied in the Ising model on a two-dimensional lattice of dimension $\beta \times L$ in zero magnetic field but $k \neq k_c$ (3.1). The partition function $Z_{PP}(\mu;\delta)$ in the scaling limit $L \to \infty$, $\mu =$ constant with periodic boundary conditions in space and time direction is given by [14]

$$Z_{PP}(\mu;\delta) = \tfrac{1}{2} \lim_{L\to\infty} \mathrm{e}^{LE_0\delta} \left[\prod_{n=0}^{L-1} \left(1 + \mathrm{e}^{-L\gamma_{2n+1}\delta}\right) + \prod_{n=0}^{L-1} \left(1 - \mathrm{e}^{-L\gamma_{2n+1}\delta}\right) + \right.$$
$$\left. \times\, \mathrm{e}^{L(E_1-E_0)\delta} \left(\prod_{n=0}^{L-1} \left(1 + \mathrm{e}^{-L\gamma_{2n}\delta}\right) + \prod_{n=0}^{L-1} \left(1 - \mathrm{e}^{-L\gamma_{2n}\delta}\right) \right) \right] \tag{5.2}$$

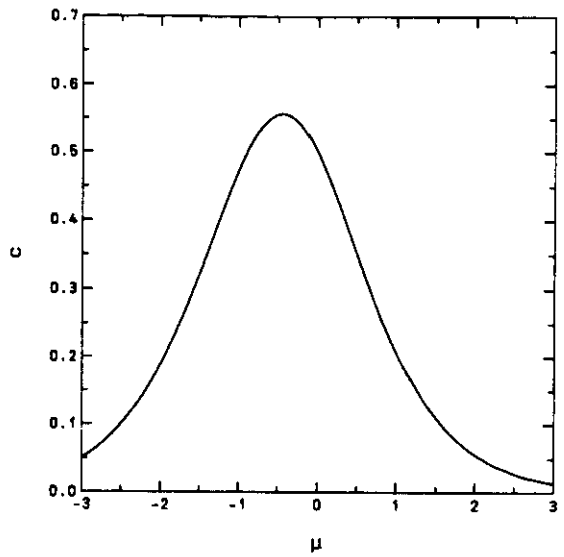

Figure 3. The universal function $C_{PP}(\mu)$ of the Ising model with thermal perturbation defined by (2.2) obtained from the partition function with periodic boundary conditions as a function of the scaling variable $\mu = L(k - k_c)$ in the large L limit (see (5.1)). At the critical point $\mu = 0$ one has $C_{PP} = c = \frac{1}{2}$ and far from the scaling region ($\mu \to \infty$) C_{PP} vanishes.

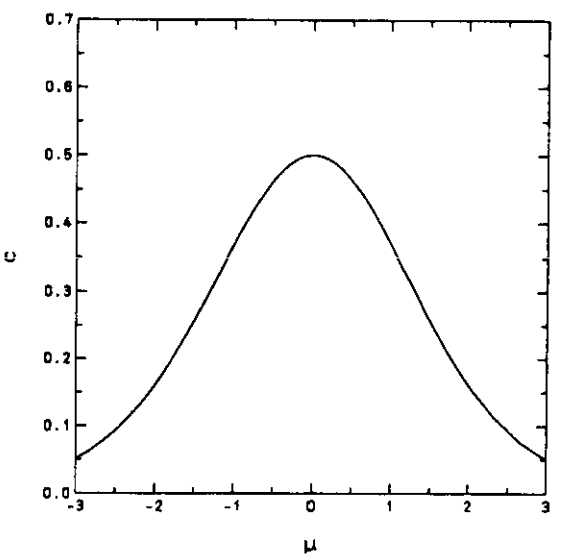

Figure 4. The universal function $\tilde{C}(\mu)$ of the Ising model with thermal perturbation defined by (2.2) obtained from the partition function (5.3) as a function of the scaling variable $\mu = L(k - k_c)$ in the large L limit (see (5.1)). As in figure 3, at the critical point $\mu = 0$ one has $\tilde{C} = c = \frac{1}{2}$ and far from the scaling region ($\mu \to \infty$) $\tilde{C}$ vanishes.

where we have chosen $\beta = L\delta$,

$$E_1 - E_0 = \tfrac{1}{2}\sum_{n=0}^{L-1}(\gamma_{2n+1} - \gamma_{2n}) \tag{5.3}$$

and

$$\gamma_n = 2\,\mathrm{arsinh}(\sqrt{4\mu^2/L^2 + \sin \pi n/2L^2})$$
$$\gamma_0 = 2\,\mathrm{arsinh}(2\mu/L). \tag{5.4}$$

From this partition function one obtains the function $C_P P(\mu)$ (2.3) shown in figure 3.

The partition function $\tilde{Z}(\mu;\delta)$ (3.6) in the limit $L \to \infty$

$$\tilde{Z}(\mu;\delta) = z^{E_0}\prod_{n=1}^{\infty}\left(1 + z^{\sqrt{(2\mu/\pi)^2+(n-1/2)^2}}\right)^2 \tag{5.5}$$

with $z = \exp -2\pi\delta$ leads to

$$\tilde{C}(\mu) = 24\sum_{n=1}^{\infty}\frac{\sqrt{(2\mu/\pi)^2 + (n-1/2)^2}}{1 + \exp\left(2\pi\sqrt{(2\mu/\pi)^2 + (n-1/2)^2}\right)} \tag{5.6}$$

shown in figure 4.

$\tilde{C}(\mu)$ is symmetric in μ, $C_{PP}(\mu)$ is not. The reason is the scaled energy gap $(E_1 - E_0)L$ which, at the critical point, is proportional to the critical exponent of the magnetization. For $\mu \to -\infty$ this scaled gap diverges, for $\mu \to \infty$, however, it vanishes. This gap contributes to Z_{PP} leading to the asymmetry in C_{PP}, but not to $\tilde{Z}$. In this asymmetry the Z_2 symmetry-breaking of the Ising model at the critical point is reflected. According to the general properties of the functions $C(g)$ both C_{PP} and $\tilde{C}$ take the value $C(0) = \frac{1}{2} = c$ at the critical point $\mu = 0$ and vanish in the limit $\mu \to \pm\infty$.

6. Conclusions

Combining the ideas of finite-size scaling and modular invariance we have studied the properties of two-dimensional statistical models near and at criticality. We defined functions $C_\alpha(g)$ of the physical parameters $\{g\}$ such as temperature or magnetic field strength characteristic for the universality class the system belongs to (see (2.3), (2.12) and (2.13)). Here the index α represents the type of boundary condition imposed on the system.

The main result of our investigation shows that for conformally invariant systems all these functions take the same value c in the critical range of coupling constants $\{g\} = \{g_c\}$, where c is the central charge of the Virasoro algebra (2.9). In the scaling region away from the critical point the C_α are continuous functions of the scaling variables μ (5.1) approaching zero when leaving the scaling region ($\mu \to \infty$). At criticality it turns out that c is given by certain sets of scaled energy gaps which are the critical exponents appearing in the corresponding statistical mechanics model.

Such sets are given by partition functions on the torus that are invariant under the modular transformation $S : \delta \to 1/\delta$ of the modular parameter δ.

For an approximate evaluation of c it turns out that not all exponents have to be determined, in fact only small subsets (e.g. only the dimensions of the relevant operators) are sufficient for a computation up to high accuracy (see examples in table 1). This is part of the magic of (2.9) making it suitable for finite-size calculations, where only low-lying excitations in the energy spectrum can be computed. Another favourable property of C_α defined by (3.8) for finite lattices is its scaling behaviour (3.9) being of the same nature as that of the critical exponents, a fact which allows good estimations from finite-size data. Finally it turns out, that c measured via (2.9) (or (3.8) respectively) is less sensitive to errors in the normalization of the Hamiltonian H than in the case of a determination using the ground state energy of H. These properties lead to excellent finite-size estimates for the central charge c even if the lattices used for the computation are extremely small. Table 3 shows data obtained from transfer matrices for only one site.

Since $C(g)$ as a function of the lattice size converges to zero if the system is not critical, in principle a measurement of C provides a determination of the critical point. So far we have not yet checked this possibility in a concrete example. As an application of our considerations we have computed the spectrum of the spin-j quantum chain (3.11) and found $c \approx 1$ independent of j, supporting the conjecture that $c = 1$. As a by-product of our investigations we showed that the modular transformation S leads to relations determining the normalization ξ in the Hamilton formalism.

Our approach to the determination of the central charge of the Virasoro algebra has led to a valuable tool in finite-size studies and given new insight into the relation between the central charge and the critical exponents. But it has also raised questions we are not yet able to answer. We would like to understand the physical meaning of the functions $C_\alpha(g)$ in the scaling region. Furthermore it would be interesting to learn whether there is any connection with the function C defined by Zamolodchikov in his famous c theorem [23] and to Cardy's recent application [4] to systems away from criticality. He shows that c can be computed in terms of certain correlation functions in the scaling region. In [24] the c theorem is reformulated by using the spectral representation for the two-point function of the stress tensor and a possible generalization to higher dimensions is discussed. In all these results critical and off-critical properties appear to be intimately related.

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